Introduction to Algebraic Number Theory
B.Math.(Hons.) Third year
First Semestral exam 2015
Instructor: Bharath Sethuraman

Remark: I will choose your **BEST SIX ANSWERS** to compute your final score. All questions have the same weight.

Remark: Minkowski's constant is $\dfrac{n!}{n^n}\left(\dfrac{4}{\pi}\right)^s \sqrt{|\mathrm{disc}(K)|}$, where K is a number field of degree n with r real and s non-real conjugate pair embeddings.

Remark: For a cubic field K with exactly one real embedding, the fundamental unit ϵ satisfies the inequality $\epsilon > \sqrt[3]{\frac{|\mathrm{disc} K|-24}{4}}$

(1) Let p be an odd prime. Prove that the field $\mathbb{Q}(\omega_p)$, where ω_p is a primitive p-th root of unity, contains the square root of $p^* := (-1)^{\frac{p-1}{2}} p$.

(2) Give with proof examples of field extensions K_1 and K_2 of $\mathbb{Q}$ and a prime $p \in \mathbb{Z}$ such that p is totally ramified in K_1 and K_2 but not in K_1K_2.

(3) Let p be an odd prime, and assume $p \equiv 1 \bmod 3$. Let ω_p be a primitive p-th root of unity, and let K be the unique subfield of $\mathbb{Q}(\omega_p)$ of degree 3 over $\mathbb{Q}$. Let $q \neq p$ be a prime in $\mathbb{Z}$. Show that q splits completely in K if and only if q is a cube mod p.

(4) If $\alpha \in K$, K a number field, is a root of a monic polynomial $f(x) \in \mathbb{Z}[x]$, and if $r \in \mathbb{Z}$ is such that $f(r) = \pm 1$, show that $\alpha - r$ is a unit of $\mathcal{O}_K$. Use this to find the fundamental unit of $\mathbb{Q}(\sqrt[3]{7})$.

(5) Let R be a Dedekind domain and I an ideal. Show that I can be generated by at most two elements.

(6) Determine with proof the class group of $\mathbb{Q}(\sqrt{-5})$, and use this to show that the equation $y^2 = x^3 - 5$ has no integer solutions.

(7) Show that $a := \sum_{n=1}^{\infty} 10^{-n!}$ is transcendental.

(8) Let K and L be number fields such that $[KL : \mathbb{Q}] = [K : \mathbb{Q}][L : \mathbb{Q}]$. If $\gcd(\mathrm{disc(K)}, \mathrm{disc}(L)) = d$, prove that $\mathcal{O}_{KL} \subseteq \frac{1}{d}\mathcal{O}_K\mathcal{O}_L$. Conclude that if $d = 1$, then if $\{\alpha_i\}$ is an integral basis for $\mathcal{O}_K$ and $\{\beta_j\}$ an integral basis for $\mathcal{O}_L$, then $\{\alpha_i\beta_j\}$ is an integral basis for $\mathcal{O}_{KL}$.